

Deviations from the $1/r^2$ Newton law due to extra dimensions

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Abstract

We systematically examine corrections to the gravitational inverse square law, which are due to compactified extra dimensions. We find the induced Yukawa-type potentials for which we calculate the strength α and range. In general the range of the Yukawa correction is given by the wavelength of the lightest Kaluza–Klein state and its strength, relative to the standard gravitational potential, by the corresponding degeneracy. In particular, when n extra dimensions are compactified on an n -torus, we find that the strength of the potential is $\alpha = 2n$, whereas the compactification on an n -sphere gives $\alpha = n + 1$. For Calabi–Yau compactifications the strength can be at most $\alpha = 20$.

1 Introduction and discussion

Recently, deviations of the inverse square law for gravity have received a lot of attention [1]–[5]. In general, these deviations are parametrized by two parameters, α and λ , corresponding to the strength, with respect to the $1/r^2$ law, and the range. Specifically, the form of the potential is (for experimental aspects, see, for instance, [5] and references therein)

$$V(r) = -\frac{G_4 M}{r} (1 + \alpha e^{-r/\lambda}) \quad , \quad (1)$$

where G_4 is the four-dimensional Newton constant and M is the mass. This form of the potential is expected to be valid for $r \gg \lambda$ and, in general, there will be more terms correcting the $1/r$ potential, which are nevertheless subdominant, as we will also see.

There are experimental bounds on the possible values of α and λ , which are represented in an α – λ diagram [1, 5], at the end of the paper. The value of λ is restricted to be at most of order 1 mm, leaving the possibility of new forces in the submillimeter regime [1, 2]. On the other hand, as depicted in the figure, there are theoretical models that can give different values of the strength α . For example, it was argued, in [1], that a Scherk–Schwarz supersymmetry-breaking mechanism at 1 TeV gives rise to a scalar radius modulus and a potential of the form (1) with $\alpha \sim 4/7$, whereas a mechanism involving the dilaton predicts $\alpha \sim 44$ [4, 1].

In this letter we systematically examine corrections to the $1/r$ gravitational potential due to extra dimensions. We consider Einstein gravity in $n + 4$ dimensions, where n is the number of extra dimensions, and we find the Newtonian limit of the theory. Then we compactify the internal n dimensions in order to obtain the four-dimensional effective gravitational potential; to leading order, this is of the form (1) with λ proportional to the inverse mass of the lightest Kaluza–Klein (KK) state and α equal to its degeneracy. We explicitly derive this result in section 2 for a general compactification manifold. In particular, for the case of an n -dimensional torus, we find, since the number of extra dimensions can be $n = 2, 3, \dots, 7$, that the strength can take the values $\alpha = 4, 6, \dots, 14$.¹ We also discuss the cases of sphere compactification, where the strength can take the values $\alpha = 3, 4, \dots, 8$ and Calabi–Yau (CY) compactification, where we argue that $\alpha \leq 20$.

Note that there exist other possibilities such as torsion, massive gravitinos, Brans–Dicke scalars etc., which are expected to produce similar corrections to the Newton law and it would be interesting to study them.

¹There cannot be only one extra dimension ($n = 1$) because the deviation from Newtonian gravity would then have been over astronomical distances [2]. The upper value in the number of extra dimensions, $n = 7$, corresponds to a compactification from the highest-dimensional consistent theory, which is eleven-dimensional supergravity.

2 Gravitational potential and extra dimensions

In this section we consider the corrections to the gravitational potential due to extra dimensions. We will first calculate the potential for the case of compactification on an n -dimensional torus and then on other spaces, including on the n -dimensional sphere and CY manifolds.

2.1 Toroidal compactification

We assume that the space-time is $(n+4)$ -dimensional, where the n extra dimensions x_i , $i = 1, 2, \dots, n$, are compactified on circles, each with radius R_i . The Newtonian limit of a $(n+4)$ -dimensional black hole will give the gravitational potential of a massive object. Since, to our knowledge, higher-dimensional black-hole solutions with some dimensions compactified are not known, we will examine the Newtonian limit of higher-dimensional gravity and we will impose compactification on the solution. Presumably, the result can also be obtained in the Newtonian limit of a, yet unknown, higher-dimensional black hole.²

The gravitational potential of a massive object with mass M at a distance $r_n = (r^2 + x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$, where $r^2 = x^2 + y^2 + z^2$ is the three-dimensional radial distance, satisfies the $(n+3)$ -dimensional Laplace equation, and it is given by

$$V_{n+4} = - \sum_{\mathbf{m} \in \mathbf{Z}} \frac{G_{n+4} M}{\left(r^2 + \sum_{i=1}^n (x_i - 2\pi R_i m_i)^2 \right)^{(n+1)/2}} . \quad (2)$$

Here, G_{n+4} is the Newton constant in $n+4$ dimensions and $\mathbf{m} = (m_1, m_2, \dots, m_n)$ is a vector in a n -dimensional lattice. This potential satisfies the appropriate boundary conditions, namely, it vanishes at spatial infinity and it is periodic in the extra n dimensions since it is invariant under the shifts $x_i \rightarrow x_i + 2\pi R_i$. For very large R_i 's only the term with $\mathbf{m} = \mathbf{0}$ survives in the sum and we recover the familiar Newton law in $n+4$ dimensions:

$$V_{n+4} \simeq - \frac{G_{n+4} M}{r_n^{n+1}} . \quad (3)$$

On the other hand, if the R_i 's are small, we may approximate the sum by an integral as

$$V_{n+4} \simeq - \frac{G_{n+4} M}{\Sigma_n} \int d^n \mathbf{x} \frac{1}{(r^2 + \mathbf{x}^2)^{(n+1)/2}} = - \frac{\Omega_n G_{n+4} M}{2 \Sigma_n} \frac{1}{r} , \quad (4)$$

where the volume of the n -dimensional torus Σ_n and that of the n -dimensional unit sphere Ω_n are given by

$$\Sigma_n = (2\pi)^n \prod_{i=1}^n R_i , \quad \Omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} . \quad (5)$$

²A periodic-black-hole solution in four space-time dimensions has been constructed in [6].

By comparing (4) with the potential in four space-time dimensions $V_4 = -\frac{G_4 M}{r}$, the four-dimensional Newton constant is identified as

$$G_4 = \frac{\Omega_n G_{n+4}}{2\Sigma_n} . \quad (6)$$

This relation, and the observed value of the four-dimensional Planck scale $G_4^{1/2} \sim 10^{-33}$ cm, leads to a unification of the Planck-scale in $n+4$ space-time dimensions (with $n \geq 2$) with the electroweak interactions scale 1 TeV (or 10^{-20} m), provided that the typical compactification radius of the circles is $R \sim 1$ mm or smaller [2], thus realizing previous proposals for large internal dimensions [7]. In turn, that suggests a novel resolution of the hierarchy problem [2, 8].

In order to discuss deviations from Newtonian gravity, we must compute the first corrections to (4). This is done by Poisson resumming (2) and we obtain³

$$\begin{aligned} V_{n+4} &= -\frac{G_{n+4}M}{\Sigma_n} \sum_{\mathbf{m} \in \mathbf{Z}} \int d^n \mathbf{x} \frac{e^{-i\tilde{\mathbf{m}} \cdot \mathbf{x}}}{\left(r^2 + \sum_{i=1}^n (x_i - 2\pi R_i m_i)^2\right)^{\frac{n+1}{2}}} \\ &= -\frac{2\Omega_{n-2}G_{n+4}M}{\Sigma_n} \sum_{\mathbf{m} \in \mathbf{Z}} e^{-i\tilde{\mathbf{m}} \cdot \mathbf{x}} \int_0^\infty d\rho \frac{\rho^{n-1}}{(r^2 + \rho^2)^{(n+1)/2}} \int_0^1 dx \cos(|\tilde{\mathbf{m}}|\rho x) (1-x^2)^{\frac{n-3}{2}} \\ &= -\frac{\Omega_{n-2}2^{n/2}\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}{2\Sigma_n} G_{n+4}M \sum_{\mathbf{m} \in \mathbf{Z}} \frac{e^{-i\tilde{\mathbf{m}} \cdot \mathbf{x}}}{|\tilde{\mathbf{m}}|^{n/2-1}} \int_0^\infty d\rho \frac{\rho^{n/2} J_{\frac{n}{2}-1}(|\tilde{\mathbf{m}}|\rho)}{(r^2 + \rho^2)^{(n+1)/2}} , \end{aligned} \quad (7)$$

where $J_{\frac{n}{2}-1}$ is the Bessel function of order $\frac{n}{2} - 1$ and $\tilde{\mathbf{m}} = \left(\frac{m_1}{R_1}, \dots, \frac{m_n}{R_n}\right)$. Note that $|\tilde{\mathbf{m}}| = \left(\frac{m_1^2}{R_1^2} + \frac{m_2^2}{R_2^2} + \dots + \frac{m_n^2}{R_n^2}\right)^{1/2}$ are the masses of the KK-modes. After performing the last integral in (7) we find

$$V_{n+4} = -\frac{G_4 M}{r} \sum_{\mathbf{m} \in \mathbf{Z}} e^{-r|\tilde{\mathbf{m}}|} e^{-i\tilde{\mathbf{m}} \cdot \mathbf{x}} , \quad (8)$$

where G_4 is defined in (6). Next we omit the internal space dependence since all point particles in the four-dimensional space-time can be taken to have $\mathbf{x} = \mathbf{0}$. Hence, the four-dimensional gravitational potential, in the presence of n extra dimensions compactified on a n -dimensional torus, is given by⁴

$$V_4 = -\frac{G_4 M}{r} \sum_{\mathbf{m} \in \mathbf{Z}} e^{-r|\tilde{\mathbf{m}}|} . \quad (9)$$

It is clear from the above expression that the Newton $1/r$ potential results from the term in the sum with $\mathbf{m} = \mathbf{0}$. The first correction to it comes from the lightest KK states. Thus, we find that the gravitational potential is approximately of the type (1), namely

$$V_4 \simeq -\frac{G_4 M}{r} (1 + 2n_0 e^{-r/R_0}) , \quad (10)$$

³The two integrals below are computed using the formulae 8.411(8) and 6.565(3) of [9].

⁴This result has been also obtained in [10] and by E. Floratos and G. Leontaris (to appear). We thank R. Rattazzi for bringing [10] into our attention.

where $1/R_0$ is the lightest KK mass and $2n_0$ is its degeneracy and n_0 is the number of equal radii, i.e. $R_1 = \dots = R_{n_0} \equiv R_0$. Thus we see that the strength equals the degeneracy of the lightest KK state and the range is its wavelength.

2.2 Compactification on other manifolds

Let us consider a space-time of the form Minkowski $\times M^n$, where M^n is an n -dimensional compact manifold; let $\{\Psi_{\mathbf{m}}\}$ be a set of functions in M^n obeying the orthogonality condition

$$\int_{M^n} \Psi_{\mathbf{n}}(\mathbf{x}) \Psi_{\mathbf{m}}^*(\mathbf{x}) = \delta_{\mathbf{n},\mathbf{m}} , \quad (11)$$

as well as the completeness relation

$$\sum_{\mathbf{m}} \Psi_{\mathbf{m}}(\mathbf{x}) \Psi_{\mathbf{m}}^*(\mathbf{x}') = \delta^{(n)}(\mathbf{x}, \mathbf{x}') . \quad (12)$$

The functions $\{\Psi_{\mathbf{n}}\}$ are eigenfunctions of the n -dimensional Laplace operator with eigenvalues $\mu_{\mathbf{m}}^2$

$$\nabla_n^2 \Psi_{\mathbf{m}} = -\mu_{\mathbf{m}}^2 \Psi_{\mathbf{m}} . \quad (13)$$

In the Newtonian limit, the gravitational potential V_{n+4} satisfies the Poisson equation in $n+3$ spatial dimensions:

$$\nabla_{n+3}^2 V_{n+4} = (n+1) \Omega_{n+2} G_{n+4} M \delta^{(n+3)}(\mathbf{x}) , \quad (14)$$

which is solved by $V_{n+4} = -\frac{G_{n+4}M}{r_n^{n+1}}$. For n compact dimensions, we may expand V_{n+4} in terms of the complete basis of eigenfunctions of the Laplace operator on M^n , $\{\Psi_{\mathbf{m}}\}$ as

$$V_{n+4} = \sum_{\mathbf{m}} \Phi_{\mathbf{m}}(r) \Psi_{\mathbf{m}}(\mathbf{x}) . \quad (15)$$

Then the $\Phi_{\mathbf{m}}$'s obey

$$\nabla_3^2 \Phi_{\mathbf{m}} - \mu_{\mathbf{m}}^2 \Phi_{\mathbf{m}} = (n+1) \Omega_{n+2} \Psi_{\mathbf{m}}^*(\mathbf{0}) G_{n+4} M \delta^{(3)}(\mathbf{x}) , \quad (16)$$

with solution

$$\Phi_{\mathbf{m}}(r) = -\frac{\Omega_n G_{n+4} M \Psi_{\mathbf{m}}^*(\mathbf{0})}{2} \frac{1}{r} e^{-\mu_{\mathbf{m}} r} , \quad (17)$$

so that (8) changes to

$$V_{n+4} = -\frac{\Omega_n G_{n+4} M}{2r} \sum_{\mathbf{m}} \Psi_{\mathbf{m}}^*(\mathbf{0}) \Psi_{\mathbf{m}}(\mathbf{x}) e^{-\mu_{\mathbf{m}} r} . \quad (18)$$

As explained before we may omit the internal space dependence and set $\mathbf{x} = \mathbf{0}$ in the above formula. Then we may further simplify it by realizing that the sum over \mathbf{m} is over all possible allowed irreducible representations of the symmetry group of the compact manifold M^n and then, for each such representation, over all representatives. However, the eigenvalue of the Laplace operator $\mu_{\mathbf{m}}$ depends only on the representation and not

on the particular representative that was used to compute it in (13). Then, from (18) we obtain the four-dimensional gravitational potential in the presence of n extra dimensions compactified on a general manifold M^n , as

$$V_4 = -\frac{G_4 M}{r} \sum_{\mathbf{m}_{\text{ir}}} d_{\mathbf{m}_{\text{ir}}} e^{-\mu_{\mathbf{m}_{\text{ir}}} r} , \quad (19)$$

where we sum up over all possible irreducible representations \mathbf{m}_{ir} , and $d_{\mathbf{m}_{\text{ir}}}$ denotes the corresponding degeneracy. The four-dimensional Newton constant G_4 is defined as in (6), where Σ_n is the volume of the compact manifold M^n . In passing from (18) to (19) we have also used the group theoretical result that the sum of $|\Psi_{\mathbf{m}_{\text{ir}}}|^2$ over all representatives of a given irreducible representation equals $d_{\mathbf{m}_{\text{ir}}}/\Sigma_n$. Using this general formula we see that, to leading order for large r , the gravitational potential is of the form (1), with range inversely proportional to the mass of the lightest KK state and strength equal to its degeneracy. The general result (19) reduces to (8) for the case of compactification on an n -torus. In that case, the symmetry group is abelian and $d_{\mathbf{m}_{\text{ir}}} = 1$.

2.2.1 Compactification on spheres

Let us illustrate these by first considering the n -dimensional sphere of radius R as our compactification manifold. A general KK state has mass and degeneracy given by [11]

$$\begin{aligned} \mu_m &= \frac{\sqrt{m(m+n-1)}}{R} , \quad m = 0, 1, \dots , \\ d_m &= \frac{(2m+n-1)(m+n-2)!}{(n-1)!m!} . \end{aligned} \quad (20)$$

Then (19) takes the form

$$V_4 = -\frac{G_4 M}{r} \sum_{m=0}^{\infty} d_m e^{-\mu_m r} , \quad (21)$$

where, the Newton constant is

$$G_4 = \frac{G_{n+4}}{2R^n} . \quad (22)$$

This potential is approximately, for large r ,

$$V_4 \simeq -\frac{G_4 M}{r} \left(1 + (n+1)e^{-\sqrt{n} r/R} \right) . \quad (23)$$

Note that the range of the induced Yukawa potential is given by the mass of the lightest KK state, whereas its strength is its degeneracy, which is $n+1$, namely the dimension of the vector representation of $SO(n+1)$. It is instructive to compare the strengths of the Yukawa-type correction for compactifications on the n -sphere and on the n -dimensional torus. For the n -sphere, the strength of the Yukawa-type correction is $\alpha = 3, 4, \dots, 8$, whereas for the n -torus $\alpha = 4, \dots, 14$. Hence, the topology of the compactification manifold of the extra dimensions seems to be hard to detect experimentally, since the strengths are comparable.

2.2.2 Compactification on Calabi–Yau manifolds

Theories with $N = 1$ supersymmetry in four-dimensions are obtained by CY compactifications in string theory. The CY manifolds are Ricci-flat Kähler manifolds with no continuous isometries and the explicit metric for them is not known. Hence, it is not possible to even attempt solving the eigenvalue equation (13). However, we may compute the degeneracy of the eigenstates using well known group theoretical results.

Typically, for CY manifolds with a (discrete) global symmetry group there exists a symmetry factor containing products of the permutation group S_n and of the cyclic group $Z_n^\ell (\equiv Z_n \times \dots \times Z_n)$.⁵ The irreducible representations of S_n are labelled by a set of n non-negative integers $\{m_i\}$, subject to the constraint [12]

$$m_1 \geq m_2 \geq \dots \geq m_n \geq 0, \quad m_1 + m_2 + \dots + m_n = n. \quad (24)$$

The dimensionality of an irreducible representation is given by

$$d_{\mathbf{m}} = \frac{n!}{h_1! h_2! \dots h_n!} \prod_{i < j}^n (h_i - h_j), \quad h_i \equiv m_i + n - i. \quad (25)$$

The lowest dimensional massless state corresponds to the solution of (24) with $m_1 = n$ and $m_2 = m_3 = \dots = m_n = 0$. It is a singlet under both S_n and Z_n^ℓ consistent with the fact that the Hodge number $h^{0,0} = 1$ for all CY spaces. The first massive state is in the lowest non-trivial representation of S_n , which corresponds to $m_1 = m_2 = \dots = m_n = 1$, and it is one-dimensional as can be seen from (25). This state is degenerate in Z_n^ℓ so that its degeneracy is at least $n\ell$. In the cases of CY manifolds with no symmetries at all, the lowest bound for the degeneracy of the first massive state is of course (we bear in mind accidental degeneracies)

$$d_{\text{lower}} = \mathcal{O}(1). \quad (26)$$

For a “maximally symmetric” isotropic quintic, the symmetry group is isomorphic to the semi-direct product of S_5 and Z_5^4 . Using this model, we obtain the upper bound for the degeneracy of the first massive state as

$$d_{\text{upper}} = 20. \quad (27)$$

Hence, the strength of the Yukawa-type correction to the inverse square law associated with the CY compactification can be at most $\alpha = 20$ which is a bit larger, but nevertheless comparable, to the values $\alpha = 12$ and $\alpha = 7$ for the torus T^6 and sphere S^6 , respectively.

Similar corrections to the Newton law are expected to arise from other sources such as torsion, massive gravitinos, Brans–Dicke scalars etc., which however have not discussed here. We have collected our results in fig. 1.

⁵A way to construct manifolds with $SU(3)$ holonomy is to start with the N -dimensional complex projective space (CP_N) and place enough constraints that reduce its complex dimensions to three. For example, for CP_4 we put $\sum_{i=1}^4 z_i^5 = 0$, for $CP_3 \times CP_3$ we put $\sum_{i=1}^3 z_i^3 = 0$, $\sum_{i=1}^3 w_i^3 = 0$ and $\sum_{i=1}^3 z_i w_i = 0$.

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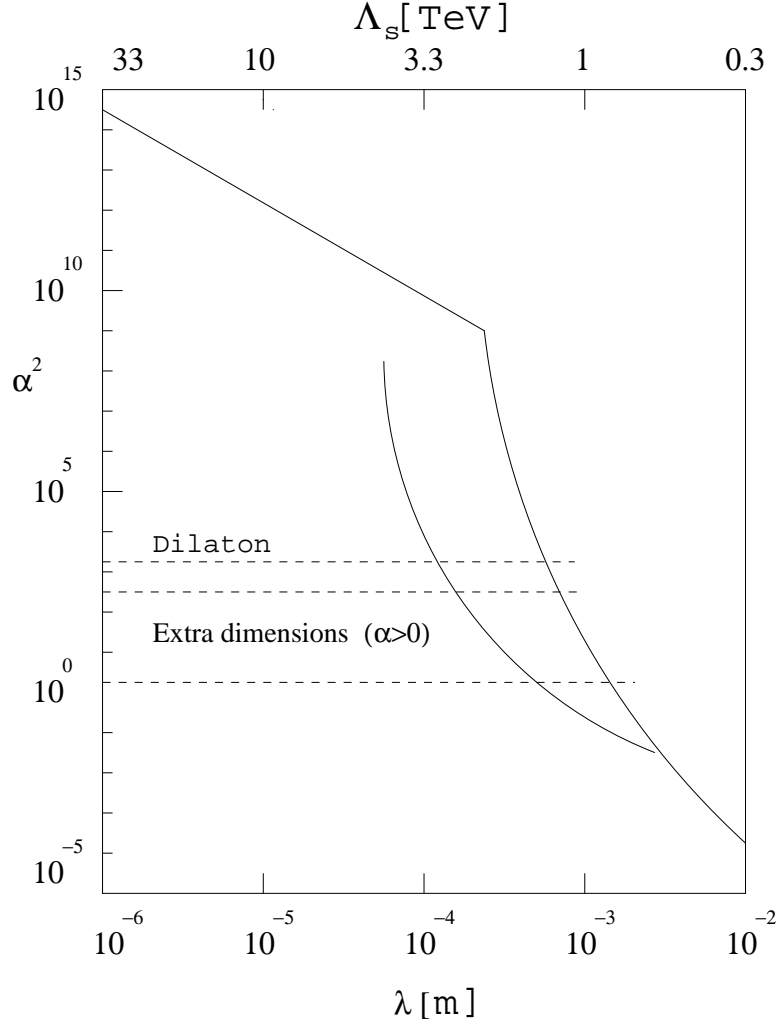


Figure 1: Plot of the α - λ diagram where the experimental bounds are indicated by solid lines according to [1, 5]. Our predictions for the strength in the case of extra dimensions range between $\alpha^2 = 1, \dots, 9, \dots, 196, \dots, 400$ as indicated by the two (---) lines. We also indicated the value $\alpha^2 \sim 2000$, which corresponds to the dilaton contribution.